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The lower domination parameters in inflation of graphs of radius 1

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Abstract

The *inflation* G_{I} of a graph G is the line graph of the subdivision of G. If G is a complete graph the equality $\mathrm{ir}(G_{\mathrm{I}}) = \gamma(G_{\mathrm{I}})$ was proved by Favaron in 1998. We conjectured that the equality holds when G is any graph of radius 1. But it turned out that it is not true. Moreover, we proved that for the class of radius 1 graphs there does not exist a better upper bound for the relation $\gamma(G_{\mathrm{I}})/\mathrm{ir}(G_{\mathrm{I}})$ then $\frac{3}{2}$. We found also a sufficient condition for the equality $\gamma(G_{\mathrm{I}}) = \mathrm{ir}(G_{\mathrm{I}})$.

Keywords: Lower domination parameters; Inflations; Claw-free graphs

We consider only simple graphs. Let $G = (V(G), E(G))$ be a graph of order $n(G)$ and diameter $\mathrm{diam}(G)$. The neighbourhood and closed neighbourhood of a vertex x of G are, respectively, denoted by $N(x)$ and $N[x]$ (with $N[x] = N(x) \cup \{x\}$). If $X \subseteq V(G)$, $G[X]$ is the subgraph induced in G by X. Let $N(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. A set X of vertices of G is *dominating* if $N[X] = V(G)$. The minimum cardinality of a dominating set is denoted by $\gamma(G)$.

If x is a vertex of a subset X of $V(G)$, the set $PN(x) = N[x] \setminus N[X \setminus \{x\}]$ is called the *X-private neighbourhood* of x and its elements are the *X-private neighbours* of x. The vertex x of X is *irredundant* in X if its X-private neighbourhood is not empty, *redundant* otherwise. The set X is *irredundant* in G if all its vertices are irredundant. If an irredundant set X is *maximal* for the inclusion, then for any vertex u which is not dominated by X there exists a non-isolated vertex y of X which is redundant in $X \cup \{u\}$. That means that u dominates every X-private neighbour of y. We say that u *annihilates* y. The minimum cardinality of a maximal irredundant set of G is denoted

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by $\mathrm{ir}(G)$. Since any minimal dominating set is a maximal irredundant set, we have for any graph G

$$\mathrm{ir}(G) \leqslant \gamma(G). \tag{1}$$

The *inflation* G_{I} of a graph G is the line graph of the subdivision of G. The subdivision of G is the graph obtained from G by replacing each edge by a path of length 2. We denote by X the clique in G_{I} corresponding to vertex x in G. If xy is an edge in G then we denote (xy) and (yx) the vertices in X and Y corresponding to this edge. An edge $(xy)(yx)$ will be a blue edge while (xy) will be a blue neighbour of (yx), the other vertices adjacent to (xy) will be the red neighbours (that are vertices in X).

Dunbar and Haynes [1] proved that for every graph H, $\mathrm{diam}(H) \leqslant 2$ if and only if $\gamma(H_{\mathrm{I}}) = n(H) - 1$.

If G is a complete graph the equality $\mathrm{ir}(G_{\mathrm{I}}) = \gamma(G_{\mathrm{I}})$ was proved in [2] by Favaron. She conjectured that the same equality holds in the case when G is a tree, which was proved by Puech [4].

It is interesting to find other classes of graphs for which this equality holds.

Let G be a *graph of radius 1* that is $\gamma(G) = 1$. Let c be a vertex which is adjacent to all other vertices of G. Then in G_{I} the vertex c is replaced by a clique C that contains $n - 1$ vertices.

We conjectured that the equality holds when G is any graph of radius 1. But it turned out that it is not true. Moreover, we proved that for the class of radius 1 graphs there does not exist a better upper bound for the relation $\gamma(G_{\mathrm{I}})/\mathrm{ir}(G_{\mathrm{I}})$ than $\frac{3}{2}$. It was shown in [3] that the bound $\frac{3}{2}$ is the best possible in the class of claw-free graphs and

$$\frac{\gamma(G_{\mathrm{I}})}{\mathrm{ir}(G_{\mathrm{I}})} < \frac{3}{2} \tag{2}$$

in the class of inflations.

Theorem 1. *Their exists a series* $\{G^k\}$ *of graphs of radius* 1, *such that*

$$\lim_{k\to\infty} \frac{\gamma(G^k_{\mathrm{I}})}{\mathrm{ir}(G^k_{\mathrm{I}})} = \frac{3}{2}.$$

Proof. For $k \geqslant 2$ graph, G^k consists of k paths (a_i, b_i, c_i) for $1 \leqslant i \leqslant k$, vertices of different paths are not adjacent, and a vertex c adjacent to all vertices of the paths, mentioned. It is clear from the definition that every G^k is a graph of radius 1. It is obvious that $\mathrm{diam}(H) \leqslant 2$ for every graph H of radius 1, hence $\gamma(G^k_{\mathrm{I}}) = n(G^k) - 1 = 3k$ by Dunbar and Haynes [1]. Consider the following set $W = \{(cb_1), (b_1a_1), (b_1c_1), \ldots, (b_ka_k), (b_kc_k)\}$ in G^k_{I}, for $k \geqslant 2$. W is irredundant, for (cb_1) is isolated in W, and the W-private neighbours of the other vertices are their blue neighbours. For an arbitrary vertex $v \in V \setminus W$ their exists of the three possibilities:

(1) $v \in C \setminus \{(cb_1)\}$, then v dominates (cb_1) and its private neighbourhood;
(2) $v \in B_i$, for some i, $1 \leqslant i \leqslant k$, in this case $N[x] \subset N[W]$;

(3) $v \in A_i$ (resp. C_i), for $1 \leqslant i \leqslant k$, then either v is a unique W-private neighbour of $(b_i a_i)$ (resp. $(b_i c_i)$) or v dominates this private neighbour.

In all cases $W \cup \{x\}$ is redundant, therefore W is a maximal irredundant set. So $\mathrm{ir}(G_{\mathrm{I}}^k) \leqslant |W| = 2k+1$.

Then using (2) we get

$$\frac{3k}{1+2k} \leqslant \frac{\gamma(G_{\mathrm{I}}^k)}{\mathrm{ir}(G_{\mathrm{I}}^k)} < \frac{3}{2}$$

and

$$\lim_{k\to\infty} \frac{\gamma(G_{\mathrm{I}}^k)}{\mathrm{ir}(G_{\mathrm{I}}^k)} = \frac{3}{2}. \qquad \square$$

Theorem 2. *Let G be a graph of radius* 1, *c be a vertex in G, adjacent to all other vertices of G, and C be the clique in G_{I} which replaces the vertex c in G. Let W be a maximal irredundant set in G_{I} with $|W| = \mathrm{ir}(G_{\mathrm{I}})$. Then $|W \cap C| \neq 1$ implies $\gamma(G_{\mathrm{I}}) = \mathrm{ir}(G_{\mathrm{I}}) = n(G) - 1$.*

Proof. We use the result of Dunbar and Haynes [1] as in Theorem 1 to get $\gamma(G_{\mathrm{I}}) = n(G) - 1$. An obvious consequence of the last equality and inequality (1) is $\mathrm{ir}(G_{\mathrm{I}}) \leqslant n(G) - 1$. Also note that $|C| = n(G) - 1$.

Assume first $|W \cap C| = 0$. We want to prove that $|W| \geqslant n(G) - 1$. If $v \in C$ then it is either a private neighbour of some vertex in W, or a vertex in $V \setminus N[W]$. Let C_1 be a set of vertices with the first property, $C_2 = C \setminus C_1$.

Next, we prove that their exists an injective mapping φ from C into W. For $v \in C_1$ let $\varphi(v)$ be the blue neighbour of v. Consider $v \in C_2$. Let B be the clique containing the blue neighbour of v. The vertex (bc) must annihilate some vertex in W, otherwise $W \cup \{(bc)\}$ is irredundant, because in this case v would be a $(W \cup \{(bc)\})$-private neighbour for (bc). Let $\varphi(v)$ be one of these vertices annihilated by (bc). $PN(\varphi(v)) \subseteq B$, otherwise $\varphi(v)$ is not annihilated by (bc).

Let $v_1, v_2 \in C_1, v_1 \neq v_2$ then $\varphi(v_1) \neq \varphi(v_2)$, otherwise $w = \varphi(v_1) = \varphi(v_2)$ would have two blue neighbours, that contradicts the simpleness of G. Now let $v_1 \in C_1, v_2 \in C_2$. The vertex v_1 is a W-private neighbour of $\varphi(v_1)$, hence W-private neighbourhood of $\varphi(v_1)$ intersects C, and W-private neighbourhood of $\varphi(v_2)$ as noted above does not intersect C, thus $\varphi(v_1) \neq \varphi(v_2)$. This time let $v_1, v_2 \in C_2, v_1 \neq v_2$. B_1 (resp. B_2) is the clique containing the blue neighbour of v_1 (resp. v_2). If $\varphi(v_1)$ is isolated in W and $\varphi(v_2)$ is not isolated in W, then $\varphi(v_1) \neq \varphi(v_2)$. If $\varphi(v_1)$ and $\varphi(v_2)$ are both isolated or both not isolated in W their W-private neighbourhoods are contained in B_1 and B_2, respectively, hence are different. Again we get $\varphi(v_1) \neq \varphi(v_2)$. We proved that for arbitrary $v_1, v_2 \in C$, $v_1 \neq v_2$ implies $\varphi(v_1) \neq \varphi(v_2)$. That is φ is an injective mapping. Hence $|\mathrm{ir}(G_{\mathrm{I}})| = |W| \geqslant |\varphi(C)| = n(G) - 1$.

Assume now $|W \cap C| \geqslant 2$. Let $v \in C \setminus W$. As $|W \cap C| \geqslant 2$ this vertex is not contained in the W-private neighbourhoods of vertices in $C \setminus W$. Let B be the clique containing the blue neighbour of v. Then B must also contain a vertex from W, otherwise $W \cup \{v\}$ is irredundant.

Again we can define φ as a mapping from C into W:

(i) for $v \in C \cap W$ we define $\varphi(v) = v$;
(ii) for $v \in C \setminus W$ we define $\varphi(v)$ as one vertex in $B \cap W$.

So if $v_1, v_2 \in C \setminus W$, with $v_1 \neq v_2$, then $\varphi(v_1)$ and $\varphi(v_2)$ lie in different red cliques. Hence $\varphi(v_1) \neq \varphi(v_2)$. As well as above φ is an injection and we have $\mathrm{ir}(G_{\mathrm{I}}) = |W| \geqslant |\varphi(C)| = n(G) - 1$. The theorem is proved. $\square$

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